PROBLEMATICAL RECREATIONS[8]

© 1966 LITTON INDUSTRIES, INC. LITHO IN U.S.A.

PROBLEMATICAL RECREATIONS[8]

Eighth in a Series

Additional copies of this booklet and others in the series are available upon request. Simply drop a card to us and ask for any or all of the following editions:

THE BEST OF PROBLEMATICAL RECREATIONS—VOL. I
compilation of booklets 1-5 in one edition

PROBLEMATICAL RECREATIONS[6]

PROBLEMATICAL RECREATIONS[7]

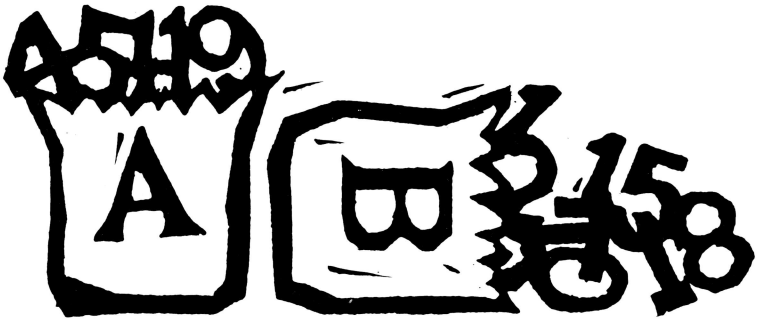
PROBLEMATICAL RECREATIONS[8]

Mail your requests to: Problematical Recreations. Litton Industries, Beverly Hills, California.

We gratefully acknowledge here the contributions from readers whose diligent efforts have consistently maintained the high quality of these mathematical diversions. We also helpfully announce that solutions to these 43 new posers will be found, as always, in the back of the booklet.

Courtesy of Litton Industries, Beverly Hills, California

1.



The set A contains the integers 0,4,5,9,11,12,13,14,19, ...
The set B contains 1,2,3,6,7,8,10,15,16,17,18, ... Place 20 and 21 in their proper sets.

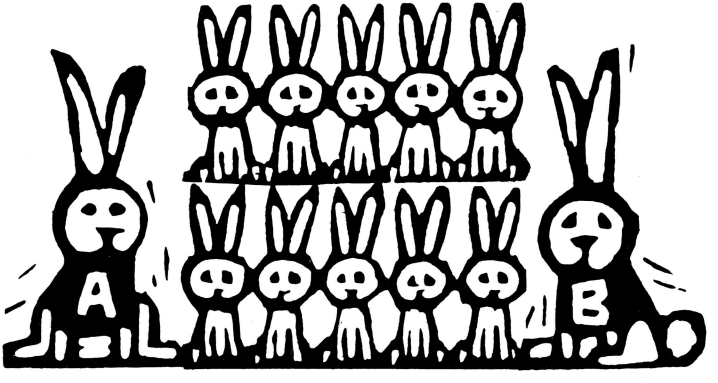
—Contributed

2.

Mr. Field, a speeder, travels on a busy highway having the same rate of traffic flow in each direction. Except for Mr. Field, the traffic is moving at the legal speed limit. Mr. Field passes one car for every nine which he meets from the opposite direction. By what percentage is he exceeding the speed limit?

—Contributed

3.



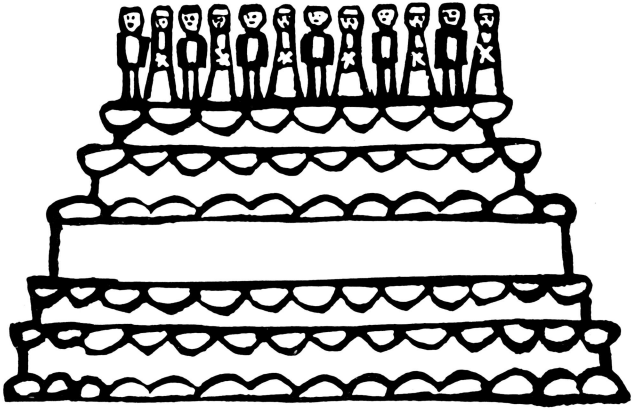
Find A and B if $7A = B$ and A and B together contain the ten digits 0 through 9 once and only once. —*Contributed*

4.

In California, automobile license plates have three letters followed by three of the digits from 0 to 9, not necessarily distinct. Is a randomly chosen car more likely to have all 6 symbols different or at least one repetition? (The zero and O are identical.)

—Contributed

5.



There are three families, each with two sons and two daughters. In how many ways can all these young people be married?

—Contributed

6.

Two octopi indulged in a friendly tentacle to tentacle wrestling match. Each managed to pin 4 of his opponent's tentacles with 4 of his own. In how many ways was this possible?

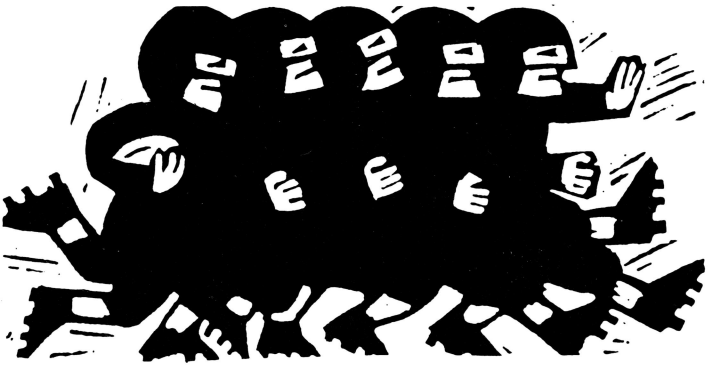
—Contributed

7.

What property is common to the sports of rowing and tug-of-war, the planet Uranus, and a clock whose hands move at the correct rate but which gives the right time only 4 times daily?

—Contributed

8.

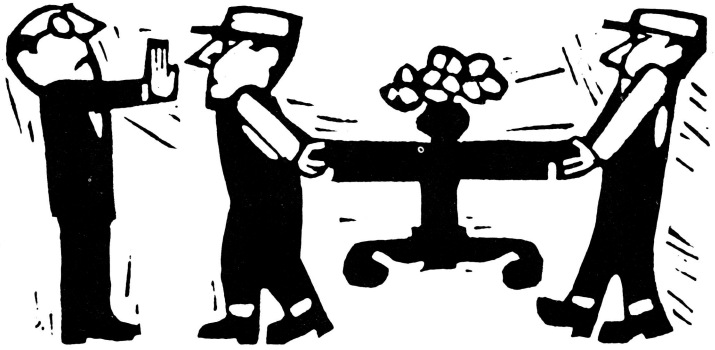


State University won their first football game of the season 17 to 0. Though they scored no safeties, they managed to score more points each quarter than they had scored in the previous quarter. What were State's quarter scores? — *Contributed*

9.

Six men decide to play Russian roulette with a six gun loaded with one cartridge. They draw for position, and afterwards, the sixth man casually suggests that instead of letting the chamber rotate in sequence, each man spin the chamber before shooting. How would this improve his chances? — *Contributed*

10.



Which is larger, $\sqrt{10} + \sqrt{17}$ or $\sqrt{53}$? No tables, please!

– Contributed

11.

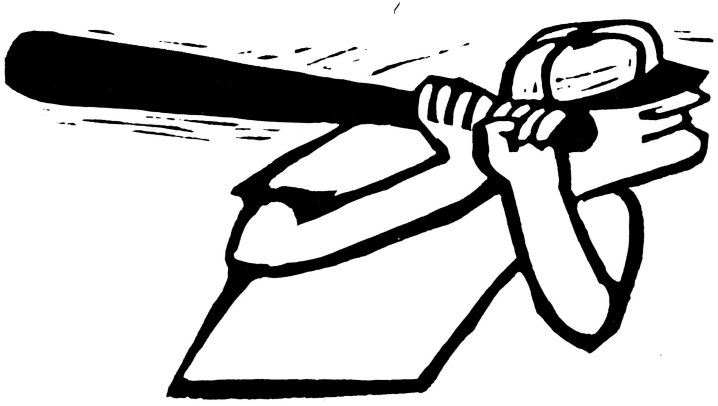
Find integers A, B, and C, positive or negative but non-zero, such that the equation $Ax^2 + Bx + C$ has roots A and B.

— Contributed

12.

At a cocktail party a man starts out with a glass of half whiskey and half soda. After each sip he adds enough soda to fill the glass again. Assuming he does this continuously (that is, in infinitesimal sips), how much whiskey has he consumed by the time he has drunk half a glass? *— Contributed*

13.

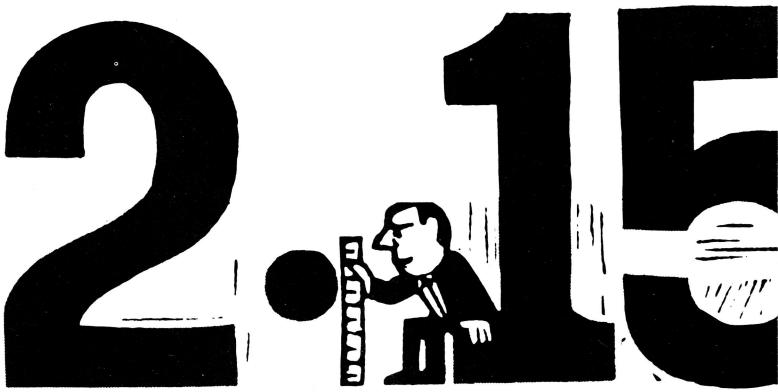


In the last major league double header of the season, Slugger Smith got 7 hits in 8 times at bat, while Joltin' Jones had 9 for 12. Which player improved his batting average more, assuming they started with identical averages? *— Contributed*

14.

In a certain code used in high-level communication in Puevigi, the two permutations of the nine positive digits 692547318 and 768415932 are code equivalents, respectively, of the words INTERVENE and EXTROVERT. Break the code and decipher 895173246. —*Contributed*

15.



In European countries the decimal point is often written a little above the line. An American, seeing a number written this way, with one digit on each side of the decimal point, assumed the numbers were to be multiplied. He obtained a two-digit number as a result, but was 14.6 off. What was the original number?

— Contributed

16.

What is the millionth term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, . . . in which each positive integer n occurs in blocks of n terms?

— Contributed

17.

A card for printed circuits has five terminals on each side. Three of the terminals on the left are to be connected to three on the right. How many different circuits are printable, i.e., do not have any printed 'wires' crossing? — *Contributed*

18.

If a coin were randomly shaken out of a certain piggy bank, its expected value would be 15 cents. If a dime had been added, the expected value would have been only 14 cents. What are the contents of the bank?

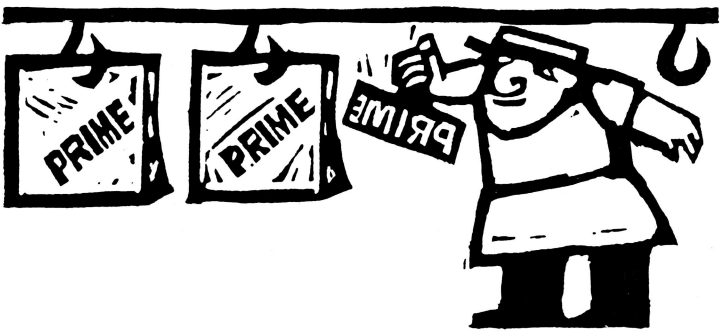
—Contributed

19.

A diaper is in the shape of a triangle with sides 24, 20 and 20 inches. The long side is wrapped around the baby's waist and overlapped two inches. The third point is brought up to the center of the overlap and pinned in place. The pin is to go through three thicknesses of material. What is the area in which the pin may be placed?

– Contributed

20.



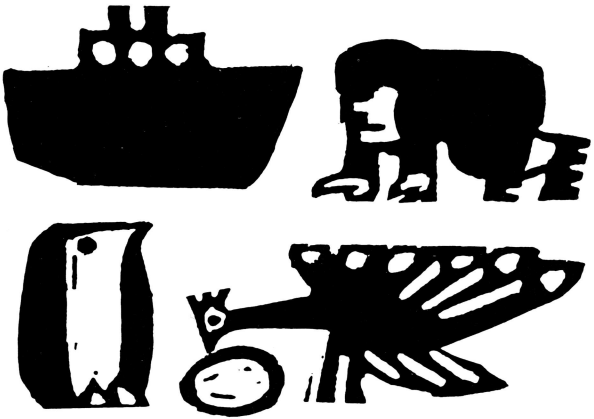
Prove that whenever P and $P^2 + 2$ are both prime, $P^2 + 4$ is also prime.

— Contributed

21.

A chemist has three large test tubes and a beaker with 54 c.c. of elixir. Using the test tubes and ingenuity only, how can he retain 50 c.c. in the beaker? — *Contributed*

22.



What property is common to Arctic penguins, peacock eggs, the Hungarian Merchant Marine, the University of Chicago football team, 19 point cribbage hands and the solution set of the equation $e^{e^x} = 1$?

– Contributed

23.

Using only the above symbols three valid equations are possible. Give yourself full credit if you get one. — *Contributed*

24.

Post-Apollo mission planners have considered the merits of orbiting synchronous (orbital period matching the rotational period of the primary) lunar satellites. At approximately what altitude with respect to the moon would such a satellite orbit?

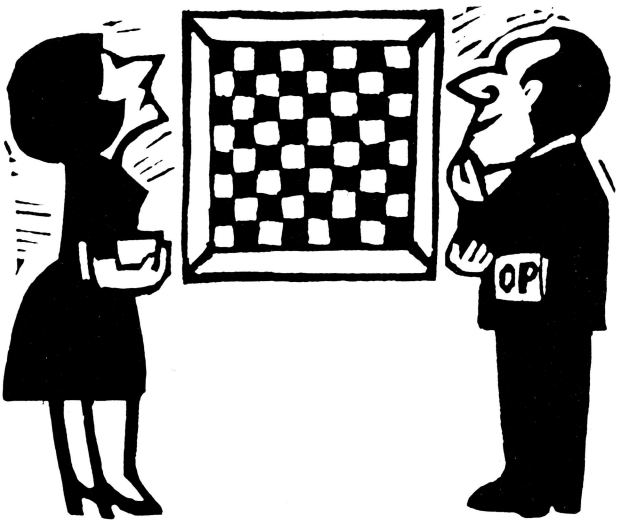
— Contributed

25.

A roulette player made 5 straight even money bets on the red. Starting with 32 chips, he bet half his current holdings each time. Red came up 3 times, black twice. In what order would he want his 3 wins to come to maximize his profit? How much would his profit be?

—Contributed

26.



How many colors are necessary for the squares of a chessboard in order to assure that a bishop cannot move from one square to another of the same color? —*Contributed*

27.

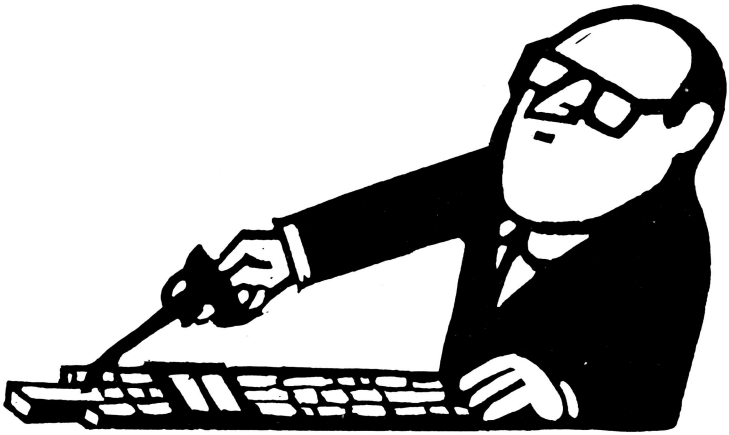
A neat computer programmer wears a clean shirt every day. If he drops off his laundry and picks up the previous week's load every Monday night, how many shirts must he own to keep him going?

—Contributed

28.

A coffee pot with a circular bottom tapers uniformly to a circular top with radius half that of the base. A mark halfway up the side says "2 cups." Where should the "3 cups" mark go?-*Contributed*

29.



A certain 3-digit number in base 10 with no repeated digits can be expressed in base R by reversing the digits. Find the smallest value of R.

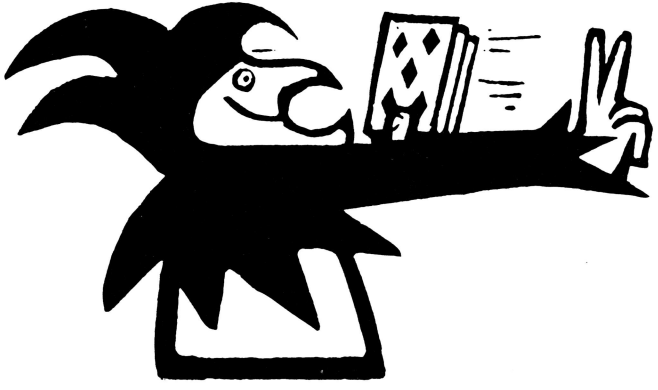
—Contributed

30.

How can seven points be placed, no three on the same line, so that every selection of three points constitutes the vertices of an isosceles triangle?

—Contributed

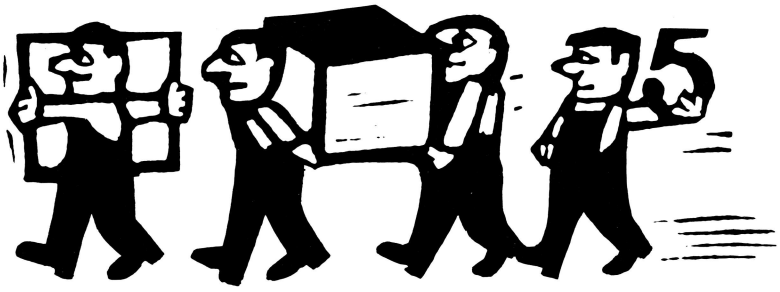
31.



A long shot poker player draws two cards to the five and six of diamonds and the joker. What are his chances of coming up with a pat hand? (straight or flush).

—Contributed

32.



One is the smallest integer which is simultaneously a perfect square, cube, and fifth power. What is the next smallest integer with this property?

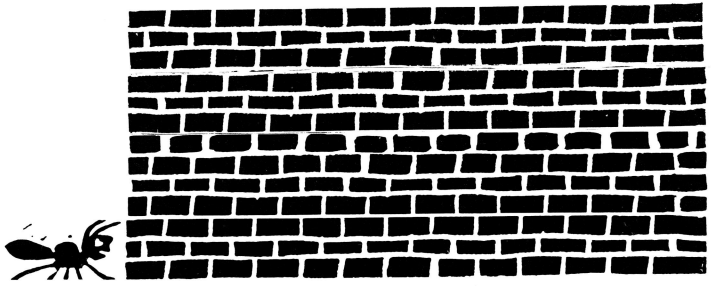
—Contributed

33.

A guidance technician celebrating a successful moon shot tipped his half full brandy glass slowly to an angle of 45° from the vertical. If the glass was spherical inside, 3″ in diameter, with a 2″ diameter hole in its top, what percent of his drink did he lose?

—Contributed

34.



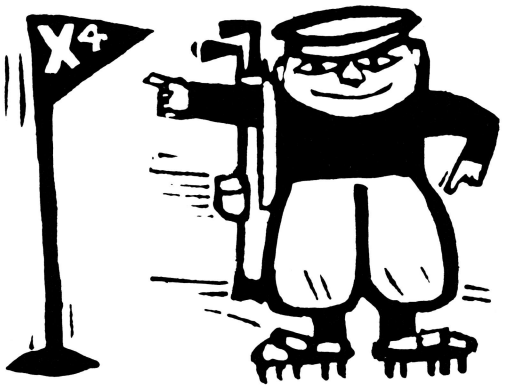
A wall is made of bricks which are twice as long as they are high. The wall is 13 courses high, with 100 bricks on the odd courses and 99 bricks plus two half bricks on the even courses. An ant starts at the lower left corner and walks in a straight line to the upper right corner. Over how many bricks does he walk?

—Contributed

35.

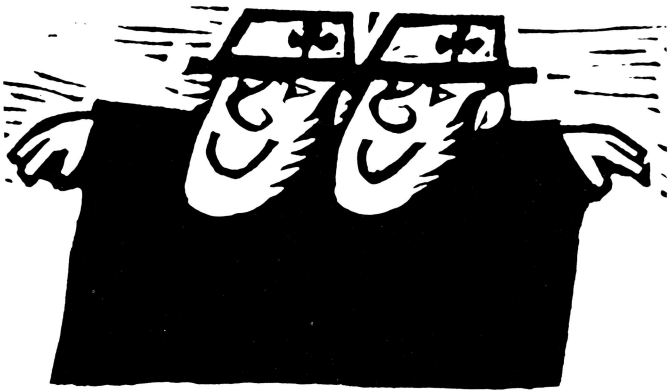
In Puevigi, the game of craps is played with a referee calling the point by adding together the six faces (three on each die) visible from his vantage point. What is the probability of making 16 the hard way? (That is, by throwing two eights.) —*Contributed*

36.



Factor the expression $x^4 + 4$. —*Contributed*

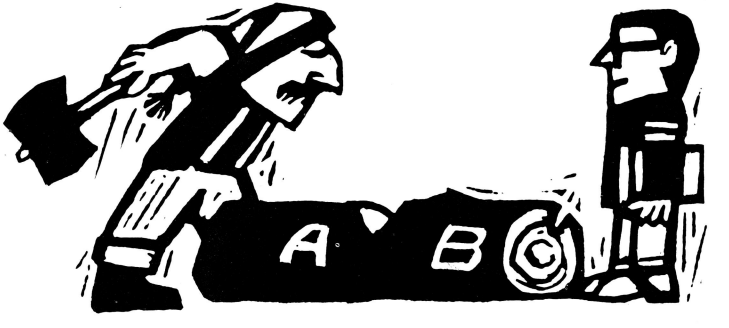
37.



Prove that at least two Irishmen have the same number of Irish friends.

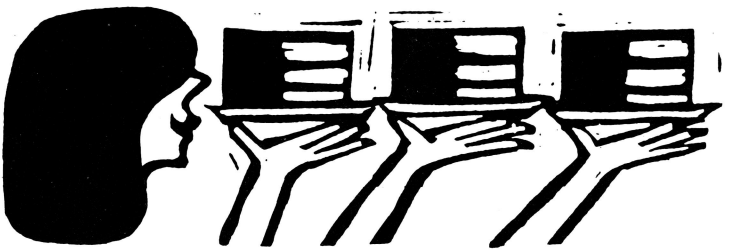
—Contributed

38.



A student, just beginning the study of logarithms, was required to evaluate an expression of the form $\frac{\log A}{\log B}$. He proceeded to cancel common factors in both numerator and denominator, (including the "factor" log), and arrived at the result $\frac{2}{3}$. Surprisingly, this was correct. What were the values of A and B? *—Contributed*

39.



A hostess plans to serve a square cake with icing on top and sides. Upon determining how many guests want cake, what method should she use to insure that each guest will receive the same amount of cake and icing? —*Contributed*

40.

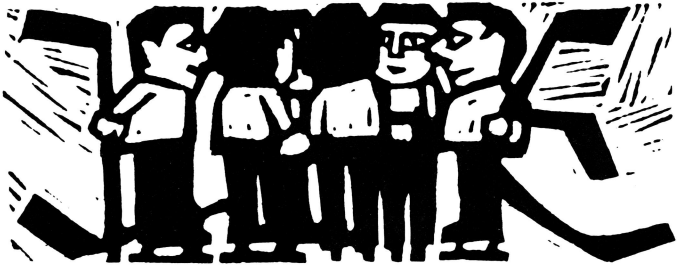
Very few people are aware of the growth pattern of Jack's beanstalk. On the first day it increased its height by ½, on the second day by ⅓, on the third day by ¼, and so on. How long did it take to achieve its maximum height (100 times its original height)?

—Contributed

41.

Tom, Dick, and Harry played a round of golf, each ending with a total of 72 strokes. Each pair competed against each other in match play (most holes won). Tom beat Dick, and Dick beat Harry. Does it follow that Tom beat Harry? —*Contributed*

42.



Six boys on a hockey team pick a captain by forming a circle and counting out until only one remains. Joe is given the option of deciding what number to count by. If he is second in the original counting order what number should he choose? —*Contributed*

43.

In a little known work, the famous geometer of Skalenos proves the following theorem: "The square of the side opposite the Fandangle is equal to the sum of the squares of the other two sides added to the product of those two sides multiplied by the square root of two." What is a Fandangle? *—Contributed*

ANSWERS

1. 20 is in A, 21 is in B. Set A contains numbers with an even (B with an odd) number of letters in their literal representation.

2. 25%. Let speed of Mr. Field = V, speed of traffic flow (i.e., speed limit) = S, average interval between cars (in either direction) = A, number of cars overtaken (same direction) per unit time = $\frac{V-S}{A}$, number of cars encountered (opposite direction) per unit time = $\frac{V+S}{A}$. Then $\frac{1}{9} = \frac{\frac{V-S}{A}}{\frac{V+S}{A}}$ and $V = \frac{5}{4}S$.

3. Both A and B must be 5-digit numbers, and A must be less than 14286. A can end only in 2, 4, 6, 7, 8 or 9. Likewise there are only 27 2-digit endings for A. These in turn place restrictions on the hundred's digit, etc., so that it is not too difficult to find the only admissible values of A and B. $7 \times 14076 = 98532$.

4. The chance that all symbols are distinct and no zero appears is
$$\frac{9}{10} \cdot \frac{8}{10} \cdot \frac{7}{10} \cdot \frac{25}{26} \cdot \frac{24}{26};$$
the chance that all symbols are distinct and one zero appears is
$$\frac{3}{10} \cdot \frac{9}{10} \cdot \frac{8}{10} \cdot \frac{25}{26} \cdot \frac{24}{26} \cdot \frac{23}{26}.$$
Thus the total probability that all symbols are different is .617.

5. There are 4 choices of wife for the oldest of the 6 boys. If his brother marries his wife's sister, 4 distinct pairings of the other people are possible. If not, 16 pairings are possible. The total is then $4\ (4 + 16) = 80$.

6. For each octopus, there are $\binom{8}{4}$ choices for the tentacles pinned and 4! permutations of these 4 tentacles with respect to the opponent's pinning tentacles. The answer is, therefore, $\binom{8}{4}^2 (4!)^2 = 2{,}822{,}400$.

7. Backwards motion.

8. Without safeties, only one answer is possible: 0, 3, 6, 8. The 3rd period score could come either from a touchdown or 2 field goals. The 4th period score had to come from a touchdown plus a 2-point conversion.

9. Normally his chance of firing the fatal bullet is $\frac{1}{6}$. With each man spinning the chamber, the probability that the first five men survive is $\left(\frac{5}{6}\right)^5$, and his chance of being shot is $\frac{1}{6}$ of that or about .067. Hence his survival probability is enhanced about .1 by spinning.

10. $(\sqrt{10} + \sqrt{17})^2 = 10 + 2\sqrt{170} + 17 > 27 + 2\sqrt{169} = 53$. Hence, $\sqrt{10} + \sqrt{17} > \sqrt{53}$.

11. We have $(x\text{-}A)(x\text{-}B) = 0$ and multiplying by A, $Ax^2 - A(A + B)x + A^2 B = 0$. Equating the coefficients of x, $B = -A(A + B)$ or $B = \frac{-A^2}{A + 1}$. Since $A + 1$ divides A^2, $A(A + 1) - A^2 = A$ is also a multiple of $A + 1$. Moreover $(A + 1) - A = 1$ is a multiple of $A + 1$. Hence $A + 1 = \pm 1$. Since $A \neq 0$, $A = -2$, $B = 4$, and $C = 16$.

12. By the time he has drunk half a glass, the respective fractions become $\frac{1}{2}e^{-1/2}$ of whiskey and $1 - \frac{1}{2}e^{-1/2}$ of soda. Since the glass was originally half full of whiskey he has drunk $\frac{1}{2} - \frac{1}{2}e^{-1/2}$ or approximately $\frac{1}{5}$ of a glass of whiskey.

13. Jones may be thought of as having had 7 for 8 followed by 2 for 4. Except in the unlikely event that he was already batting at least .500, the extra 2 for 4 cannot fail to have boosted his batting average over Smith's.

14. Inspection of the 2 code clues reveals that each letter of the word involved is also a letter of the literal form of the corresponding code digit, e.g., the first letter of EXTROVERT appears in the word SEVEN, which is the first digit of the code equivalent. "Reading through" the numbers 895173246, the only possible words are INVESTORS or INVENTORS.

15. If we assume the original number was a.b, we have $a\,b - \left(a + \frac{b}{10}\right) = 14.6$. Since a and b must be integers, b must be 4, and from $4a - a - .4 = 14.6$, we have $a = 5$. Therefore, the original number was 5.4, written as 5•4 and evaluated as 20.

16. The number k is seen to occur first at the $\left(\frac{k(k-1)}{2} + 1\right)$th position. Calling this latter expression n and solving for k, $k = \frac{1 + \sqrt{8n - 7}}{2}$. In general, by the nature of the progression, the nth term is the greatest integer $\leq$ this expression. The millionth term is, therefore, 1,414.

17. There are 10 ways of selecting the terminals on the left and 10 ways of selecting those on the right. Having selected the terminals there is only one way of connecting them so that the 'wires' do not cross, namely, keeping the order on each side the same, for if there is any reversal this will mean an intersection. There are, therefore, 10×10 or 100 different printed circuits possible.

18. Let n be the number of coins in the bank. Then $15n + 10 = 14(n + 1)$ and $n = 4$. Thus there are four coins totaling 60 cents, the only possibility being two nickels and two quarters.

19. The required area will be a diamond-shaped figure composed of two triangles with a common base, each triangle having sides $1, \frac{5}{6}$ and $\frac{5}{6}$ inches. The area of each is $\frac{1}{3}$, so that the total area in which the pin may be placed is $\frac{2}{3}$ sq. in.

20. If $P > 3$, then P is of the form $6k \pm 1$ and $P^2 + 2 = 36k^2 \pm 12k + 3$, a multiple of 3. Hence for P and $P^2 + 2$ to be prime, P must be 3, in which case $P^2 + 4 = 13$ is indeed prime. (Thus, regretfully, the problem adds no support to the famous Twin Prime Conjecture.)

21. The 10-step solution:

Beaker	Test Tubes		
54	0	0	0
0	27	27	0
27	27	0	0
27	9	9	9
45	9	0	0
45	3	3	3
51	3	0	0
51	1	1	1
52	1	1	0
52	2	0	0
50	2	2	0

The 8-step solution:

Beaker	Test Tubes		
54	0	0	0
0	27	27	0
27	27	0	0
27	9	9	9
36	18	0	0
36	6	6	6
48	6	0	0
48	2	2	2
50	2	2	0

The 6-step solution:

Beaker	Test Tubes		
54	0	0	0
0	18	18	18
36	18	0	0
36	6	6	6
48	6	0	0
48	2	2	2
50	2	2	0

Another 6-step solution:

Beaker	Test Tubes		
54	0	0	0
0	18	18	18
36	18	0	0
36	6	6	6
42	12	0	0
42	4	4	4
50	4	0	0

22. Penguins are confined to the Southern hemisphere, peahens take care of the egg-laying, Hungary is land-locked, the University of Chicago has no football team, 19-point cribbage hands are impossible, as are solutions to the equation $ee^x = 1$, since this implies $e^x = 0$, which holds for no value of x, real or complex. Thus each class is empty.

23. $6 = 3!$, $\omega^6 = 1$., $\omega^9 = 1$. where ω is the standard notation for one of the complex cube roots of one.

24. Approximately 240,000 miles, the earth-moon distance since the earth is a "synchronous satellite" of the moon.

25. The result is independent of order. With 3 wins and 2 losses, his final holdings are 32 $(3/2)^3$ $(1/2)^2$ or 27 chips. Thus there was no profit, but a loss of 5 chips.

26. It can be done with eight colors, one for each row, since a bishop move always involves a change of row. That no smaller number will do is seen from the fact that the main diagonals can have no color repeats.

27. He must pick up 7 shirts to tide him over until the following Monday. Hence he must deposit 7 shirts each Monday. Counting the shirt he wears on Monday, the required total is 15. (Note that he cannot get by with only 14 by exchanging his Monday shirt for a clean one and turning it in to the laundry as he will be caught short the following Monday.)

28. The pot is the frustum of a right circular cone with twice the height of the pot. The volume V of a cone is proportional to the cube of its height, so $\frac{V}{V-2} = \frac{1}{(3/4)^3}$, and $V = \frac{128}{37}$. Letting x= the relative distance from the apex of the "3 cups" mark, $\frac{V}{V-3} = \frac{1}{x^3}$ and $x \cong .51$. Thus the "3 cups" mark is about 2% of the way down from the top of the pot. (The capacity of the pot is exactly $3\frac{1}{37}$ cups.)

29. If we call the digits A, B and C, we have $A + 10B + 100C = C + RB + R^2A$ or $99C = (R^2 - 1)A + (R - 10)B$. R must be at least 3 in order to produce three distinct digits. Values of R from 3 to 13 yield no solution in integers A, B, C, all distinct and less than R (or 10, whichever is smaller). But $R = 14$ yields the solution $A = 4$, $B = 3$, $C = 8$. Therefore $R = 14$. $834_{10} = 438_{14}$.

30. Place five at the vertices of a regular pentagon, the sixth at the center of the pentagon, and the seventh above the center at a distance equal to the radius of the pentagon.

31. Of the $\binom{48}{2}$ different pairs of cards he could draw, there are $\binom{11}{2}$ diamond pairs and 16 pairs each of type 2-3, 2-4, 3-4, 3-7, 4-7, 4-8, 7-8, 7-9, or 8-9 which yield straights. One of each of the latter consists of 2 diamonds, i.e., has already been counted. The probability of a pat hand is therefore, $\frac{\binom{11}{2} + (15)(9)}{\binom{48}{2}}$ or about .168.

32. In the factorization of a perfect nth power, each prime factor has an exponent divisible by n. Consequently, if a number is a square, cube, and a fifth power, every prime factor has an exponent divisible by LCM (2, 3, 5) = 30. Thus the number is a perfect 30th power, and the second smallest such number is 2^{30}.

33. Zero. The angle between a circle's chord and a diameter intersecting its end is $\cos^{-1}$ C/D (C and D are the respective lengths.) By the same reasoning, if a hemisphere's volume of liquid is in the glass, it can be tipped without spilling until the angle between liquid and plane of the top is $\cos^{-1}$ C/D (C = diameter of top, and D = diameter of glass); $\cos^{-1}$ C/D = $\cos^{-1}$ 2/3, or about 48°, a safety margin of 3° even for a fuller glass. As everyone should know, guidance technicians never spill liquor.

34. The ant's path will involve 8 bricks on the first two, the middle and the last two courses and 9 bricks on the other courses. The total is, therefore, $5\times8+8\times9=$ 112 bricks.

35. The partitions of 8 into 3 numbers are (6,1,1), (5,2,1), (4,3,1), (4,2,2,), (3,3,2). Each partition involves either a duplication or 2 numbers with a sum of 7, neither of which can occur among 3 faces meeting at a corner of a correctly spotted die. Consequently, the probability of making 16 the hard way is zero.

36. By the addition and subtraction of an appropriate term, the expression becomes the difference of two squares, to wit: $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2 = (x^2 + 2x + 2)(x^2 - 2x + 2)$.

37. Suppose otherwise and assume that there are n Irishmen. The possible number of Irish friends for a given Irishman ranges from 0 to $n - 1$, and each of these possibilities must be realized in order that all n numbers be different. However, if one Irishman is a friend of all the others, no Irishman can be friendless. Thus the 0 and the $n - 1$ are mutually inconsistent.

38. We have $\frac{\log A}{\log B} = \frac{A}{B}$ where $\frac{A}{B} = \frac{2}{3}$. Hence $3 \log A = 2 \log \left(\frac{3A}{2}\right)$ from which $A = \frac{9}{4}$, $B = \frac{27}{8}$. The original expression was, therefore, $\frac{\log \frac{9}{4}}{\log \frac{27}{8}}$.

39. She lays off equally spaced marks around the perimeter of the cake, the number of marks being equal to the number of cake-eating guests, and makes a vertical cut from each mark to the center of the cake. By elementary geometry the top areas are equal, insuring equal cake volumes, and the side areas are also equal, insuring equal icing areas.

40. 198 days. Every day it increased its height by one half of its original height. In 198 days, it reproduced its height 99 times and was therefore 100 times its original height.

41. No. For example, Tom could have shot 3, 4, and 5 followed by 15 4's, Dick 4, 5, and 3 followed by 15 4's and Harry 5, 3, 4, followed by 15 4's, in which case Tom would end 1 up on Dick, Dick 1 up on Harry, and Harry 1 up on Tom.

42. Any counting interval from 1 through 9 will result in another team member's selection. Joe will be captain if he chooses 10.

43. 135°. Clearly the theorem concerns triangles and states that $a^2 = b^2 + c^2 + \sqrt{2}\,bc$, where a is the side opposite the Fandangle A. By the law of cosines, $a^2 = b^2 + c^2 - 2bc \cos A$, whence $\cos A = -\frac{\sqrt{2}}{2}$.

We hope you have enjoyed this set of mathematical challenges designed to delight your inventive faculties. We at Litton delight in the application of mathematical and scientific disciplines to the practical problems facing us in our fast-moving, technological fields of endeavor.

If you are an engineer, mathematician, or scientist with the ability to translate your skills into advanced electronic components and systems, you are invited to apply for a Litton career.

Our Plants, Laboratories, and Facilities throughout the United States and the Free World are continually expanding and creating new positions in these general areas: Business Equipment, Defense and Space Systems, Industrial Products and Services.

LITTON INDUSTRIES, Beverly Hills, California

An equal opportunity employer M/F

LITTON INDUSTRIES, INC.

Beverly Hills, California